CHAPTER 3

Vectors

Topic Covered

PHYSICAL QUANTITY

The quantity which can be measured is called physical quantity.

It is of three types

1. scalar **2.** vector **3.** tensor

1. **Scalar :** The physical quantity which can be completely described by magnitude only is called scalar. The rules for combining scalars are the rules of ordinary algebra.

   For example, mass, time, distance, speed, work, power, temperature etc.

2. **Vector :** The physical quantity which has magnitude as well as direction. Vectors must obey triangle law of addition.

   For example : displacement, velocity, acceleration, force etc.

3. **Tensor :** It is the physical quantity whose magnitude depends on direction i.e its magnitude is different along different direction.

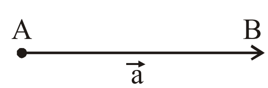
   For example : Moment of inertia, stress etc.

REPRESENTATION OF A VECTOR

A vector is generally, represented graphically or symbolically.

Graphical representation : It is represented by a line segment headed by an arrow sign as follows

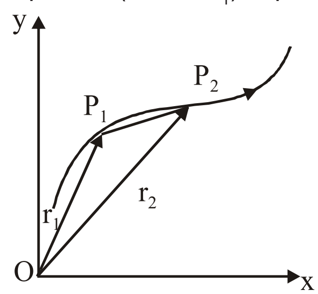
The length of the line segment represents magnitude of the vector while arrowtip indicates direction of the vectors. In this above figure, point A is called initial point, tail or origin of the vector while B is called final point, head or terminus of the vector. **Symbolic Representation :** It is generally, represented by two letters or one letter as arrow headed or bold faced as $\overrightarrow{AB}$ or $\vec{a}$ or **a**



POSITION & DISPLACEMENT VECTORS

Position vector describes the position of an object moving in a plane. In order to describe the position one require to choose a convenient point, say O as origin. Let P_1 and P_2 be the positions of the object at time t_1 and t_2, respectively. $\overrightarrow{OP_1}$ is the position vector of the object at time t_1. $\overrightarrow{OP_1} \equiv \vec{r_1}$. Point P_2 is represented by another position vector, $\overrightarrow{OP_2}$ denoted by $\vec{r_2}$. The length of the vector $\vec{r}$ represents the magnitude of the vector and its direction in the direction in which P lies as seen from O

If the object moves from P_1 to P_2, the vector $\overrightarrow{P_1P_2}$ (with tail at P_1 and tip at P_2) is called the **displacement vector** corresponding to motion from point P (at time t_1) to point P_2 (at time t_2).

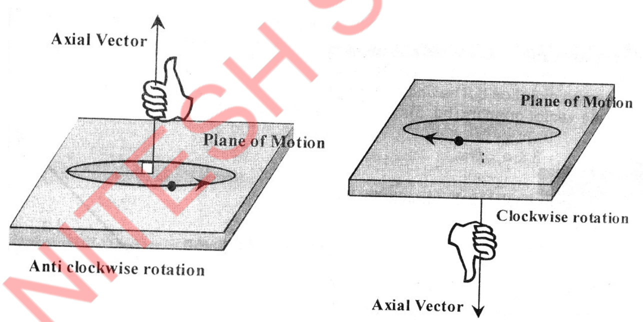


TYPES OF VECTOR

a. Polar - Vectors : have starting point (like displacement) or a point of application (like force)

b. Axial - Vectors : Rotational effects are represented by axial vectors. They are along axis of rotation, direction denoted by right hand thumb rule or right hand screw rule.

Example Angular displacement, angular velocity, torque, angular momentum.



SOME OTHER TYPES OF VECTOR

a. Negative of a vector : It has direction just opposite to given vector and have same magnitude fig.**a.**

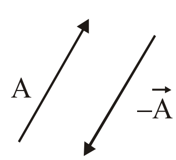
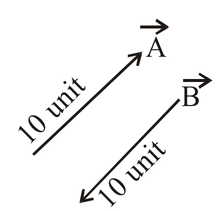


Fig (a)

Example : Let a vector $\vec{A}$. Another vector $\vec{B}$ is assumed whose magnitude is same to that of $\vec{A}$ but direction is just opposite to the vector $\vec{A}$. Then vector $\vec{B}$ is called negative vector of $\vec{A}$



$\boxed{\vec{A} - \vec{B}}$

b. Zero vector or null vector : A vector will zero magnitude having no specific direction is called zero vector fig.**b.**

i. Multiplying a vector by zero. i.e. $0(\vec{A}) = \vec{0}$

ii. By adding a negative vector to the given vector. $\vec{A} + (-\vec{A}) = \vec{0}$

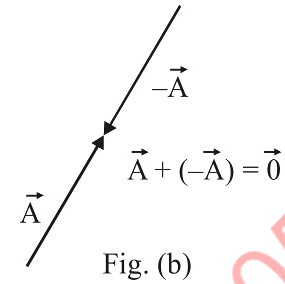


Fig. (b)

c. Equal vector : Two vectors are called equal (or equivalent) vectors if they have equal magnitude, and same direction fig.**c.** $\vec{A} = \vec{B} = \vec{C}$

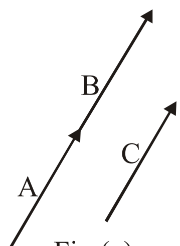


Fig (c)

d. Collinear vectors : Two vectors acting along same straight line or along parallel straight line in same direction or in opposite direction fig.**d.**

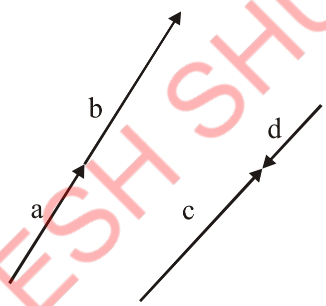


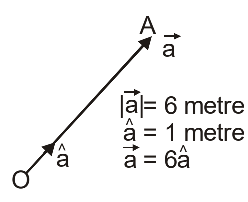
Fig. (d)

e. Coplanar vectors : Three (or more) vectors are called coplanar vectors if they lie in the same plane or are parallel to the same plane. Two (free) vectors are always coplanar.

f. Unit Vector : A vector having unit magnitude. It is used to denote the direction of a given vector.

$\vec{A} = \hat{a}.A$

$\hat{a}$ is unit vector along the direction of $\vec{A}$, $\hat{a} = \frac{\vec{A}}{|A|}$

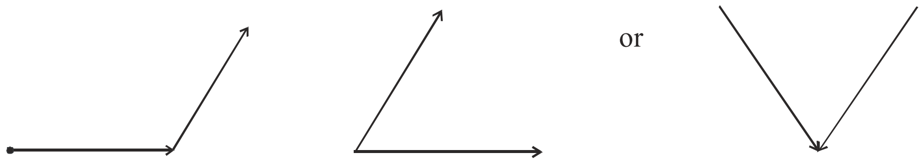


Example

As shown in figure if OA is displacement vector in any arbitrary direction then $\hat{a}$ is unit vector in the same direction.

ORDERED AND OPPOSITELY ORDERED VECTORS

When the tail of a particular vector coincides with the head of another vector they are called ordered. But when tails of both or heads of both vectors coincide they are called oppositely ordered vectors.

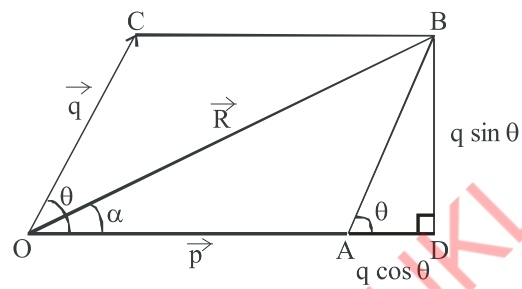


Addition of Vectors : The vector addition of two vectors is called resultant vector.

LAW OF PARALLELOGRAM

The resultant of two vectors in opposite order is given by the diagonal of the parallelogram constructed by assuming both vectors as adjacent sides.

The resultant is along $\overrightarrow{OB}$ in opposite order.



Resultant of $\vec{p}$ and $\vec{q}$ is $\vec{R}$ given by $\overline{OB}$

The length of OB = $|\vec{R}| = \sqrt{p^2 + q^2 + 2pq\cos\theta}$

where θ is angle between vectors $\vec{p}$ and $\vec{q}$. The angle between $\vec{R}$ and $\vec{p}$ is 'α' then

$$\tan\alpha = \frac{q\sin\theta}{p + q\cos\theta}$$

Proof

In ΔOBD

$$(OB)^2 = (OD)^2 + (BD)^2 = (OA + AD)^2 + (BD)^2$$

$$R^2 = p^2 + q^2 + 2pq\cos\theta \; |\vec{R}| = \sqrt{p^2 + q^2 + 2pq\cos\theta} \qquad \text{...i.}$$

$$\tan\alpha = \frac{BD}{OD}$$

$$\Rightarrow \tan\alpha = \frac{q\sin\theta}{p + q\cos\theta}$$

Special Cases

1. For parallel vectors

   $\theta = 0°$

   From equation **1.**

   $R^2 = p^2 + q^2 + 2pq\cos 0°$

   $\Rightarrow R^2 = (p^2 + q^2)^2$

   $\boxed{|\vec{R}| = |\vec{p}| + |\vec{q}|}$

2. For antiparallel vectors

$\theta = 180^0$

$R^2 = p^2 + q^2 + 2pq\cos 180°$

$\boxed{|\vec{R}| = |\vec{p}| - |\vec{q}|}$, if p > q, otherwise R = q – p

3. For $\theta = 90°$

$R^2 = p^2 + q^2$

$\boxed{|\vec{R}| = \sqrt{p^2 + q^2}}$

SOLVED EXAMPLE

Example 1. Two forces whose magnitudes an in the ratio of 3:5 give a resultant 35 N. If the angle of inclination is 60°. Find the magnitude of each force

Solution. $\frac{F_1}{F_2} = \frac{3}{5} \Rightarrow F_1 = \frac{3}{5}F_2 \Rightarrow \theta = 60°$

R = 35 N

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta°} = \sqrt{\frac{9}{25}F_2^2 + F_2^2 + 2\times\frac{3}{5}F_2^2\times\frac{1}{2}}$$

$$\Rightarrow 35 = \sqrt{\frac{49}{25}F_2^2} \Rightarrow F_2 = 35\times\frac{5}{7}$$

$F_2 = 25N$

$\therefore F_1 = 15N$

Example 2. The resultant of two forces 3p and 2p is R. If the 1st force in doubled then the resultant in also doubled. Find the angle between the two forces.

Solution. $R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$

$R^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\theta = (3p)^2 + (2p)^2 + 2(3p)(2p)\cos\theta$

$R^2 = 13p^2 + 12p^2\cos\theta$ ...**1.**

Now, $F_1 = 2 \times 3p = 6p$ then R = 2R

but $F_2 = 2p$

$\Rightarrow (2R)^2 = (6p)^2 + (2p)^2 + 2(6p)(2p)\cos\theta$

$R^2 = 10p^2 + 6p^2\cos\theta$ ...**2.**

From **1.** & **2.** we get

$10p^2 + 6p^2\cos\theta = 13p^2 + 12p^2\cos\theta$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{-1}{2}\right) = 120°$

EXERCISE

1. At what angle the two force $(F_1 + F_2)$ and $(F_1 - F_2)$ act so that the resultant is $\sqrt{(2)(F_1^2 + F_2^2)}$.

(Ans. 90°)

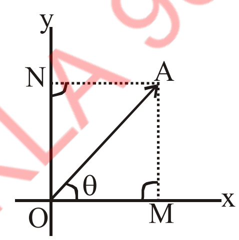
2. A boy is simultaneously given two velocities one 10 m/s due $\vec{E}$ and other 20 m/s due N – W. Calculate the resultant velocity.

[Ans. 14.74 m/s; 16°19′ west of work]

3. A wooden block of 1 kg rests on a smooth surface inclined 30° with the horizontal. Find the components of the weight (of the block) perpendicular and parallel to the plane. **(Ans. $4.9\sqrt{3}$ N; 4.9 N)**

RESOLUTION OF A VECTOR

Let a vector $\vec{A}$ makes angle θ with positive x-axis.



OM = projection of $\vec{A}$ along x-axis

OM = x component of $A = A_x$

$\Rightarrow A_x = A\cos\theta$

AM = A_y = component of $\vec{A}$ along y-axis.

$A_y = A\sin\theta$

$\vec{A}$ can be written as $\vec{A} = A_x\hat{i} + A_y\hat{j}$ in 2 D

$A^2 = A_x^2 + A_y^2 + 2A_xA_y\cos 90°$

$A^2 = A^2\cos^2\theta + A^2\sin^2\theta + 0$

$A^2 = A^2(\cos^2\theta + \sin^2\theta) = A^2$

$\Rightarrow$ Magnitude of $\vec{A} = |\vec{A}| = \sqrt{A_x^2 + A_y^2}$

In 3 D

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

Magnitude of $\vec{A} = |\vec{A}|$

$\sqrt{A_x^2 + A_y^2 + A_z^2}$

SOLVED EXAMPLE

Example 3. A bird moves with velocity 20 m/s in a direction making an angle of 60° with the eastern line and 60° with vertical upward. Represent the velocity vector in rectangular component form.

Solution : Let eastearn line be taken as x-axis, northern as y-axis and vertical upward as z-axis respectively. Let the velocity makes angle α, β, γ with axes x, y and z respectively.

Then $\alpha = 60°, \gamma = 60°$

We have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad \cos^2 60° + \cos^2\beta + \cos^2 60° = 1$

$\Rightarrow \quad \cos^2\beta = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \vec{V} = V\cos\alpha\,\hat{i} + V\cos\beta\,\hat{j} + V\cos\gamma\,\hat{k}$

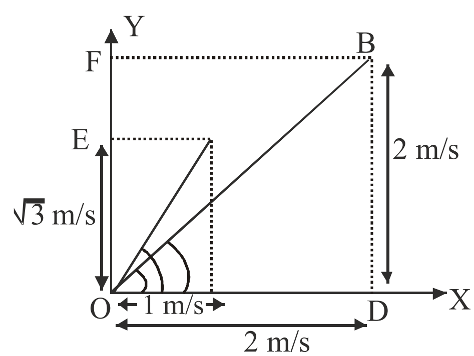
$$\boxed{\vec{V} = 10\hat{i} + 10\sqrt{2}\hat{j} + 10\hat{k}}$$

Example 4. Two billard balls are rolling on a flat table one has velocity components v_x = 1m/s, v_y = $\sqrt{3}$m/s and the other has components $v'_x = 2\text{m/s}, v'_y = 2\text{m/s}$. If both the balls starts moving from the same point, what is the angle between their paths?

Solution : 1st Ball

$OC = v_x = 1\text{m/s}$

$OE = vy = \sqrt{3}\text{ m/s}$



Resulant $OA = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{1+3} = 2\text{m/s}$

$\tan\alpha = \dfrac{AC}{OC} = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

$\therefore \quad \alpha = 60°$

2nd Balls

$v'_x = 2$ m/s

$v'_y = 2$ m/s

Resulant $OB = \sqrt{{v'_x}^2 + {v'_y}^2} = \sqrt{2+2} = 2\sqrt{2}\text{ m/s}$

$\tan\alpha' = \dfrac{BD}{OD} = \dfrac{2}{2} = 1$

$\Rightarrow \quad \alpha' = 45°$

Angle between their paths = $\alpha - \alpha' = 60° - 45° = 15°$

EXERCISE

4. A force is inclined at 30° to the horizontal. If its rectangular component in the horizontal direction is 50N, find the magnitude of the force and its vertical component.

(Ans. F = 5774 N, Vertical F = 28.8 N)

5. A car mass 1000 kg is resting on an inclined plane making an angle 30° to the horizontal. What in the weight of the car? Also find the horizontal and vertical components of the weight of the car. (g = 10 m/s^2)

6. The magnitude of the x & y components of $\vec{A}$ are 7 and 6 respectively. The magnitude sof x and y components of the $\vec{A}+\vec{B}$ an 11 and 9 respectively. Calculate the magnitude of $\vec{B}$. (**Ans.** $|\vec{B}|$ **= 5)**

SUBTRACTION OF VECTORS

Subtraction of $\vec{q}$ from $\vec{p}$ is represented as $\vec{s}=\vec{p}\times\vec{q}$

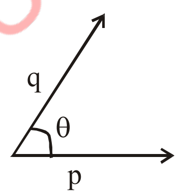
It can be written as

$\vec{s}=\vec{p}\times(-\vec{q})$ where $-\vec{q}$ is the negative of $\vec{q}$. Subtraction of $\vec{q}$ from $\vec{p}$ is the vector addition of $\vec{p}$ and $(-\vec{q})$.
It can be found by using the law of parallelogram

So, by law of parallelogram

$$|\vec{s}|=\sqrt{p^2+q^2+2pq\cos(180°-\theta)}$$

$$|\vec{s}|=\sqrt{p^2+q^2-2pq\cos\theta}$$

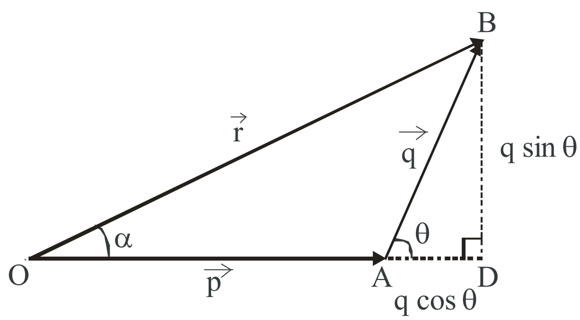


The angle between $\vec{s}$ and $\vec{p}$ is α then

$$\tan\alpha=\frac{q\sin(180°-\theta)}{p+q\cos(180°-\theta)}=\frac{q\sin\theta}{p-q\sin\theta}$$

TRIANGLE'S LAW

When two vectors are in the same order then the resultant is given by the closing side of the triangle in opposite order.



For example :

Two vectors $\vec{p}$ and $\vec{q}$ are in the same order, the resultant $\vec{r}$ is the closing side of the triangle in opposite order $\vec{p}+\vec{q}=\vec{r}$.

In ΔOBD

$(OB)^2 = (OD)^2 + (BD)^2 = (OA + AD)^2 + (BD)^2$

$r^2 = p^2 + q + 2pq\cos\theta \; |\vec{r}| = \sqrt{p^2 + q^2 + 2pq\cos\theta}$ ...**i.**

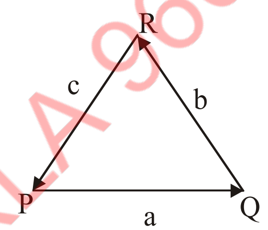
$\tan\alpha = \frac{BD}{OD}$

$\Rightarrow \tan\alpha = \frac{q\sin\theta}{p + q\cos\theta}$

SOLVED EXAMPLE

Example 5. $\vec{a}, \vec{b}, \vec{c}$ are three consecutive vectors forming a triangle, show $\vec{a} + \vec{b} + \vec{c} = 0$

Solution. Let $\overrightarrow{PQ} = \vec{a}, \overrightarrow{QR} = \vec{b}$ in the same order.



According to Δ law. resultant of $\vec{a}$ & $\vec{b} = \overrightarrow{PR}$

i.e. $\overrightarrow{PR} = \vec{a} + \vec{b}$

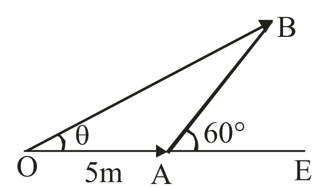
as $\overrightarrow{PR} = -\overrightarrow{RP} = -\vec{c}$

$\Rightarrow \quad -\vec{c} = \vec{a} + \vec{b}$

$\Rightarrow \quad \vec{a} + \vec{b} + c = 0$

Example 6. A boy walks $5m\vec{E}$ then turns at an angle 60° to the $\vec{N}$ of the $\vec{E}$ and walks 5m. Calculate the net displacement of the boy. Also find the direction of the displacement.

Solution. $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$



$|\overrightarrow{OB}| = \sqrt{5^2 + 5^2 + 2 \times 5 \times 5\cos 60°}$

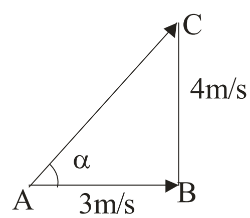
$= 5\sqrt{3} = 8.66$ m

$\tan\theta = \frac{5\sin 60°}{5 + 5\cos 60°} = 0.5773$

$\Rightarrow \theta = \tan^{-1}(0.5773) = 30°$ (with the $\vec{E}$)

Example 7. Find the resultant of two velocities, 3m/s $\vec{E}$ 4 m/s $\vec{N}$

Solution : $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$



$|\overrightarrow{AC}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 m/s$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53°8'$

EXERCISE

6. The length of second's hand of a watch is 1 cm. What is change in velocity of its tip in 15 seconds?

Ans. $\left(\frac{2\pi}{30\sqrt{3}}\ \text{cm/s}\right)$

7. A particle is moving $\vec{E}$ with velocity of 5m/s. In 10 seconds, the velocity changes to 5m/s $\vec{N}$. What is the average acceleration in this time?

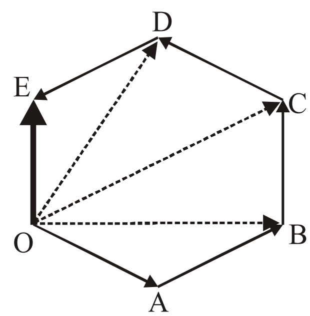
Ans. $\left(\frac{1}{\sqrt{2}}\vec{N} - \vec{W}\right)$

8. A person moves 30 m $\vec{N}$, the 20 m $\vec{E}$ and finally $30\sqrt{2}$ S – W. What is his displacemetn from the original position? **(Ans. 10 mW)**

9. A particle has a displacement of 12 m $\vec{E}$ and 5m $\vec{N}$ and then 6m vertically upwards. Find the magnitude of the sum of these displacement. **(Ans. 14.32 m)**

POLYGON'S LAW

It is the extended form of triangle's law. When many vectors are connected in the same order, the resultant is given by the closing side of the polygon in oppposite order.



For example :

Vectors $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}$ are in the same order, the closing side $\overrightarrow{OE}$ is in opposite order. So it represents the resultant of given vectors.

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$$

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$$

$$\overrightarrow{OE} = \overrightarrow{ED} + \overrightarrow{OD}$$

$$\Rightarrow \quad \overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$$

MULTIPLICATION OF A SCALAR INTO A VECTOR

When a scalar is multiplied into a vector, the result is always a vector quantity. Its magnitude is scalartimes of magnitude of the vector while direction remains same to the original vector

viz : $k(\vec{A}) = (k\vec{A})$

PRODUCT OF TWO VECTORS

The product of two vectors are of two types:

1. Scalar or dot product
2. Vector or cross product

1. Scalar or dot product : It is always a scalar quantity. It is defined as the multiplication of magnitudes of two vectors and the cosine of the least angle between given vectors. The dot product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\cdot\vec{B}$.

$$\boxed{\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta}$$

Important points regarding dot product

1. $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

   i.e. it obeys commutative laws.

2. Associative law is not defined for dot product.

3. $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

   i.e. it obeys distributive law

4. $\vec{A}\cdot\vec{A} = A^2$

5. $\vec{A}\cdot\vec{B} =$ product of magnitude of $\vec{A}$ and projection of $\vec{B}$ along $\vec{A}$.

6. $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$ as $\cos 0° = 1$

7. $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$ as $\cos 90° = 0$

8. If θ is an angle between $\vec{A}$ & $\vec{B}$ then $\dfrac{\vec{A}\cdot\vec{B}}{AB} = \cos\theta$

9. $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

   $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

   then $\vec{A}\cdot\vec{B} = A_x b_x + A_y B_y + A_z B_z$

Example 1. $W = \vec{F}\cdot\vec{S}$ **2.** $P = \vec{F}\cdot\vec{V}$ **3.** $\phi = \vec{B}\cdot\vec{A}$

SOLVED EXAMPLE

Example 8. Two vector $\vec{A} = \hat{i} + 2\hat{j} + 2\hat{k}$ $\vec{B} = \hat{i} + 3\hat{j} + 6\hat{k}$

Find **i.** Dot product **ii.** Angle between them

$A = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$ $B = \sqrt{(1)^2 + (3)^2 + (6)^2} = \sqrt{46}$

$\vec{A}.\vec{B} = (i + 2\hat{j} + 2\hat{k}).(i + 3\hat{j} + 6\hat{k})$ = (1 × 1) + (2 × 3) + (2 × 6) = 19

ii. Angle between them

$\vec{A}.\vec{B} = AB\cos\theta \Rightarrow \cos\theta = \dfrac{\vec{A}.\vec{B}}{AB}$

$\cos\theta = \dfrac{19}{3\sqrt{46}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{19}{3\sqrt{46}}\right)$

Example 9. A particle, under constant force $\hat{i}+\hat{j}-2\hat{k}$ gets displaced from point A(2, –1, 3) to B(4, 3, 2) . Find the work done by the force -

Solution. Force = $\hat{i}+\hat{j}-2\hat{k}$

dispalcement $= d = \vec{AB} = (4\hat{i}+3\hat{j}+2\hat{k})-(2\hat{i}-\hat{j}+3\hat{k}) = (2\hat{i}+4\hat{j}-\hat{k})$

work done = F.d = $(\hat{i}+\hat{j}-2\hat{k}).(2\hat{i}+4\hat{j}-\hat{k})$

= 1 x 2 + 1 x 4 + (–2) x (–1) = 2 + 4 + 2 = 8 units

Example 10. Show that the vectors **a** = $3\hat{i}-2\hat{j}+\hat{k}$, $b=\hat{i}-3\hat{j}+5\hat{k}$+, $c=2\hat{i}+\hat{j}-4\hat{k}$ form a right angled triangle.

Solution. We have **b** + **c** = $(\hat{i}-3\hat{j}+5\hat{k})+(2\hat{i}+\hat{j}-4\hat{k})=3\hat{i}-2\hat{j}+\hat{k}=a$

$\Rightarrow$ a, b, c are coplanar

Hence no two of these vectors are parallel, therfore, the given vectors form a triangle.

a . c = $(3\hat{i}-2\hat{j}+\hat{k}).(2\hat{i}+\hat{j}-4\hat{k})$ = 3 x 2 – 2 x 1 – 4 x 1 = 0

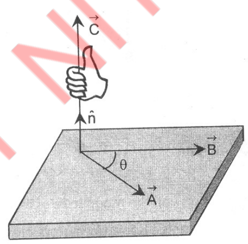
Hence the given vectors form a right angled triangle.

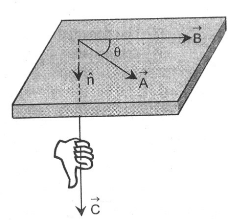
CROSS PRODUCT OR VECTOR PRODUCT OF TWO VECTORS

Cross product of $\vec{A}$ and $\vec{B}$ inclined to each other at an angle θ is defined as :

$A\,B\sin\theta\ \hat{n} = \vec{A}\times\vec{B}$ $\hat{n}$ $\perp$ to plane of $\vec{A}$ and $\vec{B}$

Direction of $\hat{n}$ is given by right hand thumb rule. Curl the fingers of your right hand from $\vec{A}$ to $\vec{B}$. Then the direction of the erect thumb will point in the direction $\vec{A}\times\vec{B}$.





Example

i. $\vec{\tau}=\vec{r}\times\vec{F}$, **ii.** $\vec{J}=\vec{r}\times\vec{p}$, **iii.** $\vec{v}=\vec{\omega}\times\vec{r}$, **iv.** $\vec{a}=\vec{\alpha}\times\vec{r}$

PROPERTIES OF VECTOR PRODUCT

a. The vector product is ‘not’ commutative i.e

$\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$

$\boxed{\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}}$

b. The vector product is distributive i.e.

$$\boxed{\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}}$$

c. The magnitude of the vector product of two vectors mutually at right angles is equal to the product of the magnitudes of the vectors.

$\vec{A} \times \vec{B} = AB \sin 90^\circ \, \hat{n} = AB \, \hat{n}, |\vec{A} \times \vec{B}| = AB$ (if $\theta = 90^\circ$)

d. The vector product of two parallel vectors is a null vector (or zero vector)

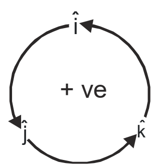
$\vec{A} \times \vec{B} = AB(\sin\theta) \, \hat{n} = \vec{0}$ or 0

e. The vector product of a vector by itself is a null vector (zero vector)

$\vec{A} \times \vec{B} = AA(\sin 0) \, \hat{n} = \vec{0}$ or 0

f. The vector product of unit orthogonal vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ have the following relations in the right handed coordinate system

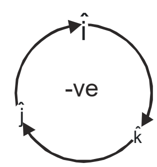
i.



$\hat{i} \times \hat{j} = \hat{k}$ $\quad \hat{j} \times \hat{i} = -\hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$ $\quad \hat{k} \times \hat{j} = -\hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$ $\quad \hat{i} \times \hat{k} = -\hat{j}$



ii. $\hat{i} \times \hat{i} = 0 \quad \hat{j} \times \hat{j} = 0 \quad \hat{k} \times \hat{k} = 0$

The magnitude of each of the vectors $\hat{i}, \hat{j}$ and $\hat{k}$ is 1 and the angle between any two of them is 90º.

Therefore, we write $\hat{i} \times \hat{j} = (1)$ **1.** sin 90º $\hat{n} = \hat{n}$, where $\hat{n}$ is a unit a vector perpendicular to the plane of $\hat{i}$ and $\hat{j}$ i.e. it is just the third unit vector $\hat{k}$.

g. The vector product of two vectors in terms of their x, y & z components can be expressed as a determinant. Let $\vec{A}$ and $\vec{B}$ be two vectors. Let us write their rectangular components

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$\vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) = (A_yB_z - B_yA_z+)\hat{i} - (A_xB_z - B_xA_z)\hat{j} + (A_xB_y - B_xA_z)\hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

SOLVED EXAMPLE

Example 11. The vector from origin to the points A and B are $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$ respectively. Find the area of

i. the triangle OAB

ii. the parallelogram formed by **OA** and **OB** as adjacent sides.

Solution. Given $\overrightarrow{OA} = \vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\overrightarrow{OB} = \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore \quad (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$= (12 - 2)\ \hat{i} - (-6 - 4)\ \hat{j} + (3 + 12)\ \hat{k}$

$= 10\hat{i} + 10\hat{j} + 15\hat{k}$

$\Rightarrow |a \times b| = \sqrt{10^2 + 10^2 + 15^2} = \sqrt{425} = 5\sqrt{17}$

i. Area of $\Delta OAB = \frac{1}{2}|a \times b| = \frac{5\sqrt{17}}{2}$ sq. units

ii. Area of parallelogram formed by OA and OB as adjacent sides = |a × b| = $5\sqrt{17}$ sq. units.

Example 12. Find a × b and b × a if

i. $a = 3\hat{k} + 4\hat{j},\ b = \hat{i} + \hat{j} - \hat{k}$

ii. a = (2, –1, 1) ; b = (3, 4, –1)

Solution. **i.** $a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 3 \\ 1 & 1 & -3 \end{vmatrix} = -7\hat{i} + 3\hat{j} - 4\hat{k}$

$b \times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 0 & 4 & 3 \end{vmatrix} = 7\hat{i} - 3\hat{j} + 4\hat{k}$

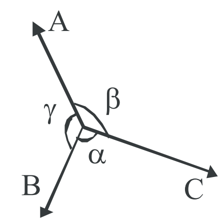
ii. $a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3\hat{i} + 5\hat{j} + 11\hat{k}$

$b \times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 3\hat{i} - 5\hat{j} - 11\hat{k}$

LAMI'S THEOREM

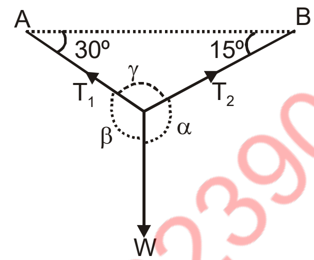
If three forces acting at a point are in equilibrium then each force is proportional to sine of the angle between the other two.

or $\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma}$



SOLVED EXAMPLE

Example 13. A rope is stretched between two poles. A 50 N boy hangs from it, as shown in fig. Find the tensions in the two parts of the rope.



Solution. In fig. $\alpha = 90° + 15° = 105°$

$\beta = 90° + 30° = 120°$

and $\gamma = 180° - (30° + 15°) = 135°$

Using Lami's Theorem, we have

$$\frac{T_1}{\sin\alpha} = \frac{T_2}{\sin\beta} = \frac{W}{\sin\gamma}$$

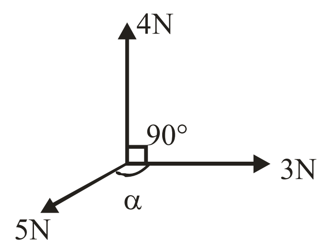
$$\therefore\ T_1 = W \times \frac{\sin\alpha}{\sin\gamma} = 50 \times \frac{\sin 105°}{\sin 135°}$$

$$= 50 \times \frac{\sin 75°}{\sin 45°} = \frac{50 \times 0.9659}{0.7071} = 68.3\ \text{N}$$

$$T_2 = \frac{W\ \sin\beta}{\sin\gamma} = \frac{50 \times \sin 120°}{\sin 135°} = \frac{50 \times \sin 60°}{\sin 45°} = \frac{50 \times 0.8660}{0.7071} = 61.24\ \text{N}$$

Example 14. Three forces, 3N, 4N, 5N are acting on a body together and the body is in equilibrium. If the angle between 3N and 4N forces is 90°, then what is the angle between 3N and 5N forces?

Solution. The forces are shown in fig. Then according to Lami's theorem



$$\frac{4}{\sin(\pi - \alpha)} = \frac{5}{\sin(\pi - 90°)}$$

Thus $\sin(\pi - \alpha) = \frac{4}{5}$

$$\pi - \alpha = \sin^{-1}\left(\frac{4}{5}\right) = 53°$$

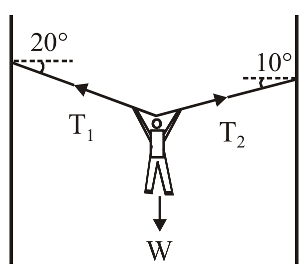
(remember the 3, 4, 5 triangle)

Thus $\alpha = 180 - 53° = 127°$

EXERCISE

10. The concurrent forces 8N, 6N and 15N are acting on an object. Is it possible to arrange the direction of forces in such a way that the object remain in equilibrium?

11. A boy of weight 100 N hangs from teh rope extending between two pipes as shown in figure. Calculate the tensions in the two parts of the rope.



SPECIAL POINTS ABOUT VECTOR

- If a vector $\vec{A}$ is multiplied by zero, we get a vector whose magnitude is zero called null vector or Zero vector.
- The unit of $n\vec{A}$ is same as that of $\vec{A}$, if n is a pure real number.
- The unit of vector does not change on being multiplied by a dimensionless scalar.
- The unit of $n\vec{A}$ is different from that of $\vec{A}$, if n is a dimensional scalar.
- The multilication of velocity vector by time gives us displacement.
- Sum of non-coplanar forces can never be zero.
- Minimum number of equal forces required for a zero resultant is two.
- Minimum number of unequal forces required for a zero resultant is three.

SOLVED EXAMPLE

Example 15. Prove that the vectors $\vec{A} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ are mutually perpendicular.

Solution : $\vec{A}.\vec{B} = (2\times 1) + (-3\times 1) + (1\times 1) = 0$

$\therefore \vec{A} \perp \vec{B}$

Example 16. Find the angle between two vectors

$\vec{A} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{B} = \hat{i} - \hat{k}$

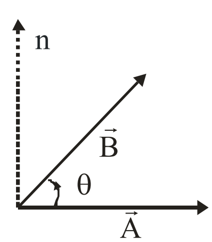
Solution : $|\vec{A}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$|\vec{B}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|} = \frac{2}{\sqrt{6}.\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \left(\frac{\pi}{6}\right)$$

2. **Cross Product of two vectors :** The cross product of two vectors $\vec{A}$ and $\vec{B}$ is represented as $\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta.\hat{n}$



Where θ is the least angle between $\vec{A}$ and $\vec{B}$ and $\hat{n}$ is the unit vector along perpendicular to the plane containing $\vec{A}\times\vec{B}$ which is given by right hand thumb rule/ screw rule.

Important points regarding cross product :

1. It obeys anticommutative law.
2. It obeys distributive law

   $$\vec{A}\times(\vec{B}+\vec{C}) = (\vec{A}\times\vec{B})+(\vec{A}\times\vec{C})$$

3. The cross product of two parallel or antiparallel vectors is always zero.
4. $\vec{A}\times\vec{B} = \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$

   or $$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$